Nonexistence of Local Self-Similar Blow-up for the 3D Incompressible Navier-Stokes Equations

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Abstract

We prove the nonexistence of local self-similar solutions of the three dimensional incompressible Navier-Stokes equations. The local self-similar solutions we consider here are different from the global self-similar solutions. The self-similar scaling is only valid in an inner core region which shrinks to a point dynamically as the time, t, approaches the singularity time, T. The solution outside the inner core region is assumed to be regular. Under the assumption that the local self-similar velocity profile converges to a limiting profile as $t \to T$ in L^p for some $p \in (3, \infty)$, we prove that such local self-similar blow-up is not possible for any finite time.

1 Introduction.

In this paper, we study the 3D incompressible Navier-Stokes equations with unit viscosity and zero external force:

$$\begin{cases} u_t + (u \cdot \nabla)u = -\nabla p + \Delta u, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0(x). \end{cases} \tag{1.1}$$

We assume that the initial condition u_0 is divergence free and $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for some $p \in (3, \infty)$.

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Many physicists and mathematicians have made a great deal of effort in understanding the physical as well as the mathematical properties of the 3D incompressible Navier-Stokes equations. One of the long standing open questions is whether the solution of the 3D Navier-Stokes equations can develop a finite time singularity from a smooth initial condition [3]. Global existence and regularity of the Navier-Stokes equations have been known in two space dimensions for a long time [7]. One of the main difficulties in obtaining the global regularity of the 3D Navier-Stokes equations is mainly due to the presence of the vortex stretching, which is absent for the 2D problem. Under suitable smallness assumption on the initial condition, the local-in-time existence and regularity results have been obtained for some time [7, 14, 9]. But these results do not give any hint on the question of global existence and regularity for the 3D Navier-Stokes.

In this paper, we prove the nonexistence of local self-similar singular solutions of the 3D Navier-Stokes equations. The local self-similar solutions we consider are very different from the global self-similar solutions considered by Leray [8]. The self-similar scaling is only valid in an inner core region which shrinks to a point dynamically as the time, t, approaches the singularity time, T. The solution outside the inner core region is assumed to be regular and does not satisfy self-similar scaling. This type of local self-similar solution is developed dynamically, and has been observed in some numerical studies. Under the assumption that the local self-similar velocity profile converges to a limiting profile as $t \to T$ in L^p for some $p \in (3, \infty)$, we prove that such local self-similar blow-up is not possible. We remark that the nonexistence of global self-similar solutions has been proved by Necas, Ruzicka and Sverak in [10]. The result of [10] was further improved by Tsai in [15].

We prove our main result by using a "Dynamic Singularity Rescaling" technique. This technique is simple but effective. Below we give a brief description of this technique. Assume that the solution of the 3D Navier-Stokes develops a local self-similar singularity at $x = 0$ at time T for the first time. A typical local self-similar singular solution is of the form

$$u(x,t) = \frac{1}{\sqrt{T-t}} U(y,t), \quad p(x,t) = \frac{1}{T-t} P(y,t), \quad y = \frac{x}{\sqrt{T-t}}, \tag{1.2}$$

for $0 \leqslant t < T$. We assume that u is smooth outside an inner core region $\{x, |x| \leqslant (T-t)^{\alpha}\}$ for some $\alpha > 0$ small. In particular, $u(x,t)$ and $p(x,t)$ are bounded for any fixed $|x| > 0$ as $t \to T$. Using this condition, we can easily show that

$$|U(y,t)| \leqslant C(T)/|y|, \quad |P(y,t)| \leqslant C(T)/|y|^2, \quad \text{for} \quad |y| \gg 1, \ \ t \leqslant T. \tag{1.3}$$

Thus, it is reasonable to assume that $U \in L^p$ for some $p \in (3, \infty)$. But the L^p norm of U may be unbounded for $0 < p \leqslant 3$.

We assume that there exists a limiting profile $\overline{U}(y) \in L^p$ as $t \to T$

$$\lim_{t \to T} \|U(t) - \overline{U}\|_{L^p} = 0, \tag{1.4}$$

for some p satisfying $3 < p < \infty$.

Next, we introduce the following rescaling in time:

$$\tau = \frac{1}{2} \log \frac{T}{T - t}, \tag{1.5}$$

for $0 \leqslant t < T$. Note that by this time rescaling, we have transformed the original Navier-Stokes equations from $[0, T)$ in t to $[0, \infty)$ in the new time variable τ. Since u is smooth for $0 < t < T$, U is smooth for $0 < \tau < \infty$. It is easy to derive the equivalent evolution equations for the rescaled velocity:

$$U_\tau + U + (y \cdot \nabla)U + 2(U \cdot \nabla)U = -2\nabla P + 2\Delta U, \tag{1.6}$$

with initial condition $U|_{\tau=0} = \sqrt{T} u_0(y/\sqrt{T})$, where U satisfies $\nabla \cdot U = 0$ for all times. The problem on the possible local self-similar blowup of the Navier-Stokes equations is now converted to the problem on the large time behavior of the rescaled equations (1.6). By assumption (1.4), we know that $U \to \overline{U}$ as $\tau \to \infty$ in L^p. We will prove that the limiting velocity profile actually satisfies the steady state equation of (1.6):

$$\overline{U} + (y \cdot \nabla)\overline{U} + 2(\overline{U} \cdot \nabla)\overline{U} = -2\nabla \overline{P} + 2\Delta \overline{U}, \tag{1.7}$$

for some $\overline{P}$. Now it follows from the result of [15] that $\overline{U} \equiv 0$, which implies that $\lim_{\tau \to \infty} \|U(\tau)\|_{L^p} = 0$ for some $p \in (3, \infty)$.

The fact that $\lim_{\tau \to \infty} \|U(\tau)\|_{L^p} = 0$ is significant because it shows that the rescaled velocity field becomes small dynamically as $\tau \to \infty$. It is easy to show that if the the solution U is small in the L^p norm at some time, τ_m, the solution must decay exponentially in τ for $\tau \geqslant \tau_m$. The exponential decay in U gives a sharp dynamic growth estimate in terms of the original velocity field. In fact, it exactly cancels the dynamic singular rescaling factor, $(\sqrt{T-t})^{-1}$, in the front of U. This gives us a uniform bound on the growth of L^p for $0 < t < T$ with $p \in (3, \infty)$, and consequently it rules out the possibility of a finite time blowup at T [11, 12, 6].

The nonexistence of local self-similar blowup of the 3D Navier-Stokes equations has some interesting implication. First, the assumption on the existence of a limiting self-similar profile, $\overline{U}$, can be verified numerically if a local self-similar blow is observed

in a computation. Secondly, this result is related to a recent existence result by one of the authors [4] for the axisymmetric 3D Navier-Stokes equations with swirl. Let v^r denote the radial component of the velocity field and $r = \sqrt{x^2+y^2}$. The result in [4] shows that if $\lim_{r\to 0}|rv^r| = 0$ holds uniformly for $0 \leqslant t \leqslant T$, then the solution of the Navier-Stokes equations is regular for $t \leqslant T$. By the well-known Caffarelli-Kohn-Nirenberg result [1], if the axisymmetric 3D Navier-Stokes equations develop a finite time singularity, the singularity must lie in the z axis. One of the most likely scenarios that would violate the condition, $\lim_{r\to 0}|rv^r| = 0$, is the local self-similar blowup of the Navier-Stokes equations. The result presented in this paper would rule out such a possibility. For more discussions regarding other aspects of the Navier-Stokes equations, we refer the reader to [7, 2, 14, 9].

The rest of the paper is organized as follows. In Section 2, we state our main theorem and present its proof. The proof is divided into four subsections. A couple of technical results are deferred to the appendices.

2 The main result and regularity analysis.

Theorem 1. *Let $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for some $p \in (3,\infty)$ and T be the first local self-similar singularity time. Assume that $U(y,t)$ defined by (1.2) converges to $\overline{U}$ in L^p as $t \to T$. Then we must have $T = +\infty$, i.e. there is no finite time local self-similar blowup for the 3D Navier-Stokes equations.*

Before we prove our main theorem, we state the following well-known $\left(L^{\tilde q}, L^{\tilde p}\right)$-estimates for the heat kernel in $\mathbb{R}^3$, $e^{-t\Delta}$, where Δ is the Laplacian operator.

$$\|e^{-t\Delta}w\|_{L^{\tilde q}} \leqslant c_0\, t^{-\left(\frac{3}{\tilde p}-\frac{3}{\tilde q}\right)/2}\|w\|_{L^{\tilde p}}, \tag{2.1}$$

$$\|\nabla e^{-t\Delta}w\|_{L^{\tilde q}} \leqslant c_0\, t^{-\left(1+\frac{3}{\tilde p}-\frac{3}{\tilde q}\right)/2}\|w\|_{L^{\tilde p}}, \tag{2.2}$$

for $1 < \tilde p \leqslant \tilde q < \infty$, c_0 depends on $\tilde p$ and $\tilde q$ only. In our analysis, we take $\tilde q = p$ and $\tilde p = p/2$. For this particular choice of $\tilde p$ and $\tilde q$, we can choose a constant, c_0, such that the above two inequalities hold. Throughout the paper, we will use c_0 and c_1 to denote various constants that do not depend on the individual functions, and use C_j $(j = 1,2)$ to denote various constants that depend on the initial condition, u_0. We also define

$$\gamma = 3/p. \tag{2.3}$$

Since $3 < p < \infty$, we have $0 < \gamma < 1$.

Proof of Theorem 1.

We will prove the theorem by contradiction. Suppose that $T < +\infty$. This means that the solution u to problem (1.1) develops a singularity at $t = T$ for the first time, but u is the unique smooth solution of (1.1) for $0 < t < T$ and is bounded in L^p.

We will divide the proof into four steps, which are given in the following four subsections.

2.1 Dynamic singularity rescaling and a priori estimates.

We make the following dynamic singularity rescaling of the 3D Navier-Stoke equations:

$$\begin{cases} \tau = \dfrac{1}{2}\log\dfrac{T}{T-t}, \quad y = \dfrac{x}{\sqrt{T-t}} \\ u(x,t) = \dfrac{1}{\sqrt{T-t}}U(y,\tau), \\ p(x,t) = \dfrac{1}{T-t}P(y,\tau), \quad \text{for } \ 0 \leqslant t < T. \end{cases} \tag{2.4}$$

Note that with this dynamic singularity rescaling, we transform the time interval from $[0, T)$ in the original time variable t to $[0, \infty)$ in the rescaled time variable τ. It is easy to derive an evolution equation for the rescale velocity field:

$$\begin{cases} U_\tau + U + (y\cdot\nabla)U + 2(U\cdot\nabla)U = -2\nabla P + 2\Delta U, \\ \nabla\cdot U = 0, \\ U|_{\tau=0} = \sqrt{T}u_0\left(x\right). \end{cases} \tag{2.5}$$

Note that since $u(x, t)$ is the unique smooth solution of the Navier-Stokes equations (1.1) for $0 < t < T$, $U(x, \tau)$ is the unique smooth solution of the rescaled Navier-Stokes equations (2.5) for $0 < \tau < \infty$.

Let $\phi(y) = (\phi_1, \phi_2, \phi_3)$ be a smooth, compactly supported, divergence free vector field in $\mathbb{R}^3$ and $\psi(\tau)$ be a smooth, compactly supported test function in $(0, 1)$ satisfying $\int_0^1 \psi(\tau)d\tau = 1$. Multiplying (2.5) by $\psi(\tau - n)\phi(y)$ and integrating over $\mathbb{R}^3 \times [n, n+1]$ for some $n > 0$, we obtain after integration by parts

$$\begin{aligned} &\int_n^{n+1}\int_{\mathbb{R}^3}\left(-\psi_\tau\phi\cdot U + \psi\phi\cdot U - \psi\nabla\cdot(\phi\otimes y)\cdot U - 2\psi\nabla\phi\cdot(U\otimes U)\right)dyd\tau \\ &= 2\int_n^{n+1}\int_{\mathbb{R}^3}\psi\Delta\phi\cdot Udyd\tau, \end{aligned} \tag{2.6}$$

where ψ is evaluated at $\tau - n$.

By assumption (1.4), we have

$$\lim_{\tau\to\infty} \|U(\tau) - \overline{U}\|_{L^p} = 0, \tag{2.7}$$

for some $p > 3$. Thus $\|U(\tau)\|_{L^p}$ is bounded for τ sufficiently large, and $\|\overline{U}\|_{L^p}$ is also bounded. Let $U(\tau) = \overline{U} + R(\tau)$. By (2.7), we have $\lim_{\tau\to\infty} \|R(\tau)\|_{L^p} = 0$. Substituting $U(\tau) = \overline{U} + R(\tau)$ into (2.6) and letting $n \to \infty$, we will show that all the terms involving R will go to zero as $n \to \infty$. It is sufficient to prove this for the nonlinear term:

$$\int_n^{n+1} \int_{\mathbb{R}^3} \psi \nabla\phi \cdot (R \otimes R) dy d\tau.$$

Let $q = p/(p-2) > 1$. Then we have $2/p + 1/q = 1$. Using the Hölder inequality, we obtain

$$\begin{aligned} |\int_n^{n+1} \int_{\mathbb{R}^3} \psi \nabla\phi \cdot (R \otimes R) dy d\tau| &\leqslant C \sup_{n\leqslant\tau\leqslant n+1} \int_{\mathbb{R}^3} |\nabla\phi||R|^2 dy \\ &\leqslant C\|\nabla\phi\|_{L^q} \sup_{n\leqslant\tau\leqslant n+1} \|R(\tau)\|_{L^p}^2 \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Other terms can be proved similarly. Therefore, by letting $n \to \infty$, we get

$$\begin{aligned} &-\left(\int_0^1 \psi_\tau(\tau) d\tau\right) \int_{\mathbb{R}^3} \phi(y)\overline{U}(y) dy \\ &+\left(\int_0^1 \psi(\tau) d\tau\right) \left(\int_{\mathbb{R}^3} \left(\phi \cdot \overline{U} - \nabla \cdot (\phi \otimes y) \cdot \overline{U} - 2\nabla\phi \cdot (\overline{U} \otimes \overline{U})\right) dy\right) \\ &= 2\left(\int_0^1 \psi(\tau) d\tau\right) \left(\int_{\mathbb{R}^3} \Delta\phi \cdot \overline{U} dy\right). \end{aligned} \tag{2.8}$$

Since ψ has compact support in $[0, 1]$, we conclude that

$$\int_0^1 \psi_\tau(\tau) d\tau = 0.$$

Moreover, we have $\int_0^1 \psi(\tau) d\tau = 1$ by assumption on ψ. Thus, we obtain

$$\int_{\mathbb{R}^3} \left(\phi \cdot \overline{U} - \nabla \cdot (\phi \otimes y) \cdot \overline{U} - 2\nabla\phi \cdot (\overline{U} \otimes \overline{U}) - 2\Delta\phi \cdot \overline{U}\right) dy = 0. \tag{2.9}$$

Thus, $\overline{U}$ is a weak solution of the steady state rescaled Navier-Stokes equations:

$$\overline{U} + (y \cdot \nabla)\overline{U} + 2(\overline{U} \cdot \nabla)\overline{U} = -2\nabla \overline{P} + 2\Delta \overline{U}, \tag{2.10}$$

with $\nabla \cdot \overline{U} = 0$. Let R_j be a Riesz operator with Fourier symbol $\xi_j/|\xi|$. One can easily modify the proof of Lemma 3.1 of [10] to show that $\overline{P} = R_j R_k(\overline{U}_j \overline{U}_k)$.

Since $\overline{U} \in L^p$ for some $p \in (3, \infty)$, we can apply Theorem 1 of [15] to conclude that $\overline{U} \equiv 0$. As a result, we obtain the following *a priori* decay estimate for $\|U(\tau)\|_{L^p}$.

Lemma 1. *The solution $U(x,\tau)$ of the rescaled Navier-Stokes equations (2.5) satisfies the following decay estimate:*

$$\lim_{\tau \to \infty} \|U(\tau)\|_{L^p} = 0. \tag{2.11}$$

For the purpose of our later analysis, we will choose a τ_m large enough to satisfy the following inequality:

$$2c_0^2\, c_1 \|U(\tau_m)\|_{L^p} \leqslant \frac{1}{6}, \tag{2.12}$$

where the constant c_1 is defined by

$$c_1 = (\frac{2}{1-\gamma} + \frac{1}{2})\left(1 - e^{-2}\right)^{-\frac{1+\gamma}{2}}. \tag{2.13}$$

The reason for such a choice of τ_m will become clear later in our analysis.

2.2 Dynamic decay estimates for the rescaled equations.

In this subsection, we will perform estimates for the rescaled Navier-Stokes equations starting from $\tau = \tau_m$ with the initial value, $U(x, \tau_m)$:

$$\begin{cases} V_\tau + V + (y \cdot \nabla)V + 2(V \cdot \nabla)V = -2\nabla P + 2\Delta V \\ \nabla \cdot V = 0 \\ V|_{\tau=0}(x) \equiv V_0(x) = U(x, \tau_m), \end{cases} \tag{2.14}$$

where τ_m is defined by (2.12)-(2.13). Since $U(x,\tau)$ is the unique smooth solution of the rescaled Navier-Stokes equations (2.5) for $0 < \tau < \infty$, the function $V(x,\tau)$ defined by

$$V(x,\tau) = U(x, \tau + \tau_m), \quad \text{for} \;\; \tau \geqslant 0, \tag{2.15}$$

is the unique smooth solution of (2.14).

Next, we perform estimates for the linearized operator in (2.14)

$$\frac{\partial V}{\partial \tau} + V + (y \cdot \nabla_y)V - 2\Delta_y V = 0, \tag{2.16}$$

with initial value $V|_{\tau=0} = V_0$.

Let $y = e^{\tau}\tilde{y}$ and $\tilde{V}(\tilde{y}, \tau) \equiv V(y, \tau)$. Then we have

$$\frac{\partial \tilde{V}}{\partial \tau} + \tilde{V} - 2e^{-2\tau}\Delta_{\tilde{y}}\tilde{V} = 0, \tag{2.17}$$

with initial value $\tilde{V}|_{\tau=0} = V_0$.

Taking the Fourier transform of (2.17), we get

$$\frac{\partial \widehat{\tilde{V}}}{\partial \tau} + \widehat{\tilde{V}} + 2e^{-2\tau}|\xi|^2\widehat{\tilde{V}} = 0, \tag{2.18}$$

where the Fourier transformation is defined as $\widehat{f}(\xi) \equiv \int f(x)e^{-2\pi i x\cdot\xi}dx$. Equation (2.18) can be written as

$$\frac{\partial}{\partial \tau}\left(e^{\tau + 2|\xi|^2\int_0^\tau e^{-2s}ds}\widehat{\tilde{V}}(\tau)\right) = 0. \tag{2.19}$$

Integrating from 0 to τ, we get

$$\widehat{\tilde{V}}(\tau) = e^{-\tau - |\xi|^2\left(1-e^{-2\tau}\right)}\widehat{V_0}. \tag{2.20}$$

Using the explicit formula of the Fourier transform of a Gaussian in three space dimensions (see, e.g. [13])

$$\widehat{e^{-\pi\alpha^2|x|^2}} = \frac{1}{\alpha^3}e^{-\pi|\xi|^2/\alpha^2}, \tag{2.21}$$

with $\alpha^2 = \left(\frac{\pi}{1-e^{-2\tau}}\right)$, we obtain

$$\mathcal{F}^{-1}\left(e^{-|\xi|^2(1-e^{-2\tau})}\right) = \left(\frac{\pi}{1-e^{-2\tau}}\right)^{\frac{3}{2}} e^{-\pi^2|x|^2/(1-e^{-2\tau})}, \tag{2.22}$$

where $\mathcal{F}^{-1}f(x) \equiv \int f(\xi)e^{2\pi i x\cdot\xi}d\xi$ is the inverse Fourier transformation. Therefore, we have

$$\tilde{V}(\tilde{y}, \tau) = e^{-\tau}\left(\frac{\pi}{1-e^{-2\tau}}\right)^{\frac{3}{2}} \int V_0(\tilde{x})\left(e^{-\pi^2|\tilde{y}-\tilde{x}|^2/(1-e^{-2\tau})}\right)d\tilde{x}. \tag{2.23}$$

Denote by $e^{-\tau A}$ the solution operator of the linearized equations (2.17). Define

$$t_0(\tau) = (1 - e^{-2\tau}), \tag{2.24}$$

and denote Δ as the Laplacian operator, then we have

$$e^{-\tau A}V_0 = \tilde{V}(\tilde{y},\tau) = e^{-\tau}\left(e^{-t_0(\tau)\Delta}V_0\right). \tag{2.25}$$

Define the following bilinear operator:

$$F(U,V) = 2\left(1 - \nabla(-\Delta)^{-1}\nabla\cdot\right)\nabla\cdot(U\otimes V). \tag{2.26}$$

In particular, if we set $V = U$, we have

$$\begin{aligned} F(U,U) &= 2\left(\nabla\cdot(U\otimes U) - \nabla(-\Delta)^{-1}\nabla\cdot\nabla\cdot(U\otimes U)\right) \\ &= 2\left(U\cdot\nabla U + \nabla P\right). \end{aligned} \tag{2.27}$$

The rescaled 3D Navier-Stokes equations (2.14) can be converted into the following integral equation:

$$V(\tau) = e^{-\tau A}V_0 - \int_0^\tau e^{-(\tau-s)A}F(U,U)(s)ds. \tag{2.28}$$

To solve the integral equation (2.28), we construct a successive approximation, $V^{(n)}$, using the following iterative scheme (see [5]): $V^{(0)} = e^{-\tau A}V_0$,

$$V^{(n+1)} = V^{(0)} - G(V^{(n)}, V^{(n)}), \quad n \geqslant 0, \tag{2.29}$$

where the bilinear operator $G(U,V)$ is defined as follows:

$$G(U,V) = \int_0^\tau e^{-(\tau-s)A}F(U,V)(s)ds. \tag{2.30}$$

To establish the convergence of the approximate solution sequence, $V^{(n)}$, we need to use the following lemma, which follows from (2.25) and the well-known (L^q, L^p)-estimates (2.1)-(2.2) for the heat kernel.

Lemma 2. *Let $V \in L^{\tilde{p}}$ for $1 < \tilde{p} \leqslant \tilde{q} < \infty$. We have*

$$\|e^{-\tau A}V\|_{L^{\tilde{q}}} \leqslant c_0\, e^{-(1-3/\tilde{q})\tau} t_0(\tau)^{-\left(\frac{3}{\tilde{p}}-\frac{3}{\tilde{q}}\right)/2}\|V\|_{L^{\tilde{p}}}, \tag{2.31}$$

$$\|\nabla e^{-\tau A}V\|_{L^{\tilde{q}}} \leqslant c_0\, e^{-(2-3/\tilde{q})\tau} t_0(\tau)^{-\left(1+\frac{3}{\tilde{p}}-\frac{3}{\tilde{q}}\right)/2}\|V\|_{L^{\tilde{p}}}. \tag{2.32}$$

The lemma can be proved easily by noting that the heat kernel actually acts on the variable $\tilde{y}$ through the function $\tilde{V}(\tilde{y},\tau)$ and $\tilde{y}=e^{-\tau}y$. Thus we lose a factor $e^{3\tau/\tilde{q}}$ when we estimate the $L^{\tilde{q}}$ norm by changing variables from y to $\tilde{y}$, but we gain a factor of $e^{-\tau}$ when we differentiate with respect to y.

Applying (2.31) with $\tilde{p}=\tilde{q}=p$, we obtain

$$\|V^{(0)}\|_{L^p}(\tau)=\|e^{-\tau A}V_0\|_{L^p}(\tau)\leqslant c_0e^{-(1-\gamma)\tau}\|V_0\|_{L^p}, \tag{2.33}$$

where $\gamma=3/p$. To estimate $\|G(U,V)\|_{L^p}$, we use (2.32) with $\tilde{q}=p$ and $\tilde{p}=p/2$:

$$\|G(U,V)\|_{L^p}(\tau)\leqslant 2c_0\int_0^\tau e^{-(2-\gamma)(\tau-s)}t_0(\tau-s)^{-\frac{1+\gamma}{2}}\|U\|_{L^p}(s)\|V\|_{L^p}(s)ds, \tag{2.34}$$

where we have used the Hölder inequality $\|U\otimes V\|_{L^{p/2}}\leqslant\|U\|_{L^p}\|V\|_{L^p}$ and the fact that $(-\Delta)^{-1}\nabla\cdot\nabla\cdot$ is a Rietz operator of degree zero, which is a bounded operator from L^p to L^p. In particular, we obtain by setting $V=U$ that

$$\|G(U,U)\|_{L^p}(\tau)\leqslant 2c_0\int_0^\tau e^{-(2-\gamma)(\tau-s)}t_0(\tau-s)^{-\frac{1+\gamma}{2}}\|U\|^2_{L^p}(s)ds\ . \tag{2.35}$$

Now, applying (2.33) and (2.35) to the iterative scheme (2.29), we get

$$\|V^{(n+1)}\|_{L^p}(\tau)\leqslant c_0e^{-(1-\gamma)\tau}\|V_0\|_{L^p}+2c_0\int_0^\tau e^{-(2-\gamma)(\tau-s)}t_0\left(\tau-s\right)^{-\frac{1+\gamma}{2}}\|V^{(n)}\|^2_{L^p}(s)ds. \tag{2.36}$$

Define

$$K_n=\sup_{0\leqslant\tau<\infty}\|e^{(1-\gamma)\tau}V^{(n)}(\tau)\|_{L^p}. \tag{2.37}$$

Multiplying (2.36) by $e^{(1-\gamma)\tau}$ on both sides and using (2.37), we obtain

$$e^{(1-\gamma)\tau}\|V^{(n+1)}\|_{L^p}(\tau)\leqslant c_0\|V_0\|_{L^p}+2c_0e^{-\tau}K_n^2\int_0^\tau e^{\gamma s}\left(1-e^{-2(\tau-s)}\right)^{-\frac{1+\gamma}{2}}ds. \tag{2.38}$$

In Appendix II, we will prove that

$$e^{-\tau}\int_0^\tau e^{\gamma s}\left(1-e^{-2(\tau-s)}\right)^{-\frac{1+\gamma}{2}}ds\leqslant c_1,\quad \text{for all}\ \ \tau\geqslant 0, \tag{2.39}$$

where c_1 is defined in (2.13). Now, take the supremum of the both sides of (2.38) for all $\tau\geqslant 0$, we obtain the following recurrence inequalities:

$$K_{n+1}\leqslant K_0+MK_n^2,\quad \text{for}\ \ n\geqslant 0, \tag{2.40}$$

with $K_n|_{n=0}=K_0$, where

$$M=2c_0c_1,\quad K_0=c_0\|V_0\|_{L^p}. \tag{2.41}$$

We will prove the following lemma in Appendix I.

Lemma 3. *Let K_0 and M be two positive constants satisfying*

$$K_0 M \leqslant \frac{1}{6}, \tag{2.42}$$

then there exists a positive constant $K_{\max}$, such that

$$K_n \leqslant K_{\max}, \qquad \text{for all } \ n \geqslant 1 \tag{2.43}$$

holds for the recurrence sequence K_n satisfying (2.40). Moreover the upper bound $K_{\max}$ satisfies

$$2MK_{\max} \leqslant \frac{1}{2}. \tag{2.44}$$

Recall that in (2.12), we have chosen τ_m such that

$$2c_0^2 c_1 \|U(\tau_m)\|_{L^p} \leqslant 1/6 \ . \tag{2.45}$$

Therefore, we have

$$K_0 M \leqslant 2c_0^2 \ c_1 \ \|U(\tau_m)\|_{L^p} \leqslant \frac{1}{6}. \tag{2.46}$$

Thus, for our choice of τ_m defined in (2.12), the recurrence sequence K_n has an upper bound $K_{\max}$ for all n. That is

$$\|V^{(n)}\|_{L^p}(\tau) \leqslant K_{\max} e^{-(1-\gamma)\tau}, \quad \text{for} \quad n \geq 1. \tag{2.47}$$

2.3 Convergence of the approximate solution sequence.

In this subsection, we will establish the convergence of the approximate solution sequence, $\{V^{(n)}\}$, and study the property of its limiting solution. We will first show that the approximate solution sequence $\{V^{(n)}\}$ is a Cauchy sequence in L^p. By subtracting (2.29) with index n from that with index $n-1$, we obtain

$$\begin{aligned}
&\|V^{(n+1)} - V^{(n)}\|_{L^p} \\
&= \|G(V^{(n)}, V^{(n)}) - G(V^{(n-1)}, V^{(n-1)})\|_{L^p} \\
&= \|G(V^{(n)}, V^{(n)} - V^{(n-1)}) + G(V^{(n)} - V^{(n-1)}, V^{(n-1)})\|_{L^p}.
\end{aligned}$$

Using (2.34), (2.47) and (2.39), we obtain

$$
\begin{aligned}
& e^{(1-\gamma)\tau}\|V^{(n+1)}-V^{(n)}\|_{L^p}(\tau) \\
& \leqslant 2c_0 e^{(1-\gamma)\tau}\int_0^\tau e^{-(2-\gamma)(\tau-s)} t_0\,(\tau-s)^{-\frac{1+\gamma}{2}}\left(\|V^{(n)}\|_{L^p}+\|V^{(n-1)}\|_{L^p}\right)\|V^{(n)}-V^{(n-1)}\|_{L^p}(s)ds \\
& \leqslant 4c_0 K_{\max} e^{-\tau}\int_0^\tau e^{\gamma s} t_0\,(\tau-s)^{-\frac{1+\gamma}{2}}\,ds\left(\sup_{0\leqslant s<\infty} e^{(1-\gamma)s}\|V^{(n)}-V^{(n-1)}\|_{L^p}(s)\right) \\
& \leqslant 4c_0 c_1 K_{\max}\sup_{0\leqslant s<\infty} e^{(1-\gamma)s}\|V^{(n)}-V^{(n-1)}\|_{L^p}(s) \\
& \leqslant \frac{1}{2}\sup_{0\leqslant s<\infty} e^{(1-\gamma)s}\|V^{(n)}-V^{(n-1)}\|_{L^p}(s),
\end{aligned}
$$

where we have used (2.41) and (2.44) in deriving the last inequality. Taking the supremum on the left hand side would yield

$$
\sup_{0\leqslant\tau<\infty} e^{(1-\gamma)\tau}\|V^{(n+1)}-V^{(n)}\|_{L^p}(\tau)\leqslant\frac{1}{2}\sup_{0\leqslant\tau<\infty} e^{(1-\gamma)\tau}\|V^{(n)}-V^{(n-1)}\|_{L^p}(\tau), \quad (2.48)
$$

which implies

$$
\sup_{0\leqslant\tau<\infty} e^{(1-\gamma)\tau}\|V^{(n+m)}-V^{(n)}\|_{L^p}(\tau)\le C_1\left(\frac{1}{2}\right)^n, \quad \text{for any } n,m\ge 1, \quad (2.49)
$$

where C_1 depends on $V^{(0)}$ only. Thus $\{V^{(n)}\}$ is a Cauchy sequence in $\mathrm{BC}\left([0,\infty);L^p\left(\mathbb{R}^3\right)\right)$. Here $\mathrm{BC}\left([0,\infty);L^p\left(\mathbb{R}^3\right)\right)$ denotes the class of bounded and continuous function from $[0,\infty)$ to $L^p(\mathbb{R}^3)$. As a result, we have proved that $V^{(n)}(\tau)$ converges uniformly to a limiting function $\overline{V}(\tau)$ in $\mathrm{BC}\left([0,\infty);L^p\left(\mathbb{R}^3\right)\right)$. Taking the limit $n\to\infty$ in (2.47), we obtain

$$
\|\overline{V}\|_{L^p}(\tau)\leqslant K_{\max}e^{-(1-\gamma)\tau}. \quad (2.50)
$$

Next, we will show that $\overline{V}$ is a solution of the integral equation (2.28). To this end, we define $R^{(n)}(x,\tau)\equiv V^{(n)}(x,\tau)-\overline{V}(x,\tau)$. We have just shown that

$$
\sup_{0\leqslant\tau<\infty} e^{(1-\gamma)\tau}\|R^{(n)}\|_{L^p}(\tau)=\sup_{0\leqslant\tau<\infty} e^{(1-\gamma)\tau}\|V^{(n)}-\overline{V}\|_{L^p}(\tau)\to 0, \quad (2.51)
$$

as $n\to\infty$. Now substituting $V^{(n)}=\overline{V}+R^{(n)}$ into the iterative scheme (2.29) and using the bilinearity of operator $G(U,V)$, we get

$$
\overline{V}-V^{(0)}+G(\overline{V},\overline{V})=-(R^{(n+1)}+G(R^{(n)},\overline{V})+G(\overline{V},R^{(n)})+G(R^{(n)},R^{(n)})). \quad (2.52)
$$

We will prove that the error terms on the right hand side of (2.52) tend to zero uniformly for all $\tau \geqslant 0$. It is obvious that $\|R^{(n+1)}\|_{L^p} \to 0$ uniformly as $n \to \infty$ from (2.51).

To show that the error terms which are linear in $R^{(n)}$ tend to zero uniformly, we use (2.34) and the *a priori* bound on $\overline{V}$ given by (2.50). Specifically, we have

$$
\begin{aligned}
&\|G(R^{(n)}, \overline{V}) + G(\overline{V}, R^{(n)})\|_{L^p}(\tau) \\
&\leqslant 4c_0 \int_0^\tau e^{-(2-\gamma)(\tau-s)} t_0\,(\tau - s)^{-\frac{1+\gamma}{2}} \|R^{(n)}\|_{L^p}(s)\|\overline{V}\|_{L^p}(s) ds \\
&\leqslant 4c_0 K_{\max} e^{-(1-\gamma)\tau} e^{-\tau} \int_0^\tau e^{\gamma s} t_0\,(\tau - s)^{-\frac{1+\gamma}{2}}\, ds \left(\sup_{0 \leqslant s < \infty} e^{(1-\gamma)s} \|R^{(n)}\|_{L^p}(s) \right) \\
&\leqslant 4c_0 c_1 e^{-(1-\gamma)\tau} K_{\max} \sup_{0 \leqslant s < \infty} e^{(1-\gamma)s} \|R^{(n)}\|_{L^p}(s) \\
&\leqslant \sup_{0 \leqslant s < \infty} e^{(1-\gamma)s} \|R^{(n)}\|_{L^p}(s) \to 0,
\end{aligned}
$$

uniformly for all τ as $n \to \infty$, where we have used $M = 2c_0c_1$ and (2.44).

To show that the nonlinear error term $G(R^{(n)}, R^{(n)})$ also tends to zero uniformly, we note that the *a priori* bounds on $V^{(n)}$ and $\overline{V}$ also provide the following *a priori* bound for $R^{(n)}$:

$$
\|R^{(n)}\|_{L^p}(\tau) \leqslant 2K_{\max} e^{-(1-\gamma)\tau}, \quad \text{for} \quad n \geq 1. \tag{2.53}
$$

Using (2.53) and applying the same argument as above, we can prove that

$$
\|G(R^{(n)}, R^{(n)})\|_{L^p}(\tau) \leqslant \sup_{0 \leqslant s < \infty} e^{(1-\gamma)s} \|R^{(n)}\|_{L^p} \to 0,
$$

uniformly for $0 \leqslant \tau < \infty$ as $n \to \infty$.

Now, passing the limit $n \to \infty$ in the L^p norm, we obtain

$$
\overline{V}(\tau) = V^{(0)} - G(\overline{V}, \overline{V}), \quad \text{for all} \ \ \tau \geqslant 0, \tag{2.54}
$$

which shows that $\overline{V}$ is a solution of the integral equation (2.28), satisfying the decay property (2.50).

2.4 The non-blowup estimates in the original variables.

In this subsection, we will complete the regularity analysis in the original physical variable. By the uniqueness of strong solutions in L^p with $p > 3$, we have

$$
\|\overline{V}\|_{L^p}(\tau) = \|U\|_{L^p}(\tau + \tau_m), \quad \text{for} \ \ 0 \leqslant \tau < \infty. \tag{2.55}
$$

Now we can use the decay estimate for $\overline{V}$ in (2.50) to obtain a decay estimate for U, which in turn will rule out the possibility of a finite time singularity for the 3D Navier-Stokes equations.

Using (2.50) and (2.55), we immediately obtain a decay estimate for U:

$$\|U\|_{L^p}(\tau) \leqslant K_{\max} e^{-(1-\gamma)(\tau-\tau_m)}, \qquad \text{for } \tau \geqslant \tau_m. \tag{2.56}$$

This proves the following decay estimate for U.

Lemma 4. *The solution $U(x,\tau)$ of rescaled Navier-Stokes equations (2.5) with τ_m defined by (2.12) has a uniform decay rate in τ as follows:*

$$\|U\|_{L^p}(\tau) \leqslant K_{\max} e^{-(1-\gamma)(\tau-\tau_m)}, \qquad \text{for } \tau \geqslant \tau_m. \tag{2.57}$$

Substituting the relation

$$u(x,t) = \frac{1}{\sqrt{T-t}} U(y,\tau) \tag{2.58}$$

into (2.57), we obtain for $t_m \leqslant t < T$ with $t_m = T\left(1 - e^{-2\tau_m}\right)$,

$$\begin{aligned}
\|u\|_{L^p}(t) &= \frac{(T-t)^{\gamma/2}}{(T-t)^{1/2}} \|U\|_{L^p}(\tau) \\
&\leqslant \frac{K_{\max}}{(T-t)^{(1-\gamma)/2}} e^{-(1-\gamma)(\tau-\tau_m)} = \frac{K_{\max} e^{(1-\gamma)\tau_m}}{(T-t)^{(1-\gamma)/2}} e^{-(1-\gamma)\tau} \\
&= \frac{K_{\max} e^{(1-\gamma)\tau_m}}{(T-t)^{(1-\gamma)/2}} \left(\frac{T-t}{T}\right)^{(1-\gamma)/2} \\
&\leqslant \frac{K_{\max} e^{(1-\gamma)\tau_m}}{T^{(1-\gamma)/2}}, \qquad \text{for} \quad t_m \leqslant t < T.
\end{aligned} \tag{2.59}$$

Since $u_0 \in L^p$ for some $p \in (3, \infty)$, it is easy to show that there is a local-in-time smooth solution whose L^p norm is bounded [5] (This can also be proved directly by using the same iterative scheme applied to the original Navier-Stokes equations for a short time). Moreover, since T is the first singularity time, we conclude that u is smooth for $0 < t \leqslant t_m < T$ and has a bounded L^p norm for $t \leqslant t_m$. Thus, $\|u\|_{L^p}(t)$ is uniformly bounded for $0 \leqslant t < T$.

Now, we can apply the so-called Ladyzhenskaya-Prodi-Serrin condition (see [6], [11] and [12]), which is also known as the $L^{p,q}$ criteria. The so-called $L^{p,q}$ criteria state that if a suitable weak solution of (1.1) satisfies

$$u \in L^q\left([0,T); L^p\left(\mathbb{R}^3\right)\right) \tag{2.60}$$

with

$$\frac{3}{p} + \frac{2}{q} \leqslant 1, \qquad p \in [3, \infty], \tag{2.61}$$

then u is a smooth solution of the 3D Navier-Stokes equation up to $t = T$. In our case, we have obtained a uniform bound in L^p for u with $p \in (3, \infty)$ for $0 \leqslant t < T$. Thus the $L^{p,q}$ criterion is satisfied with $q = \infty$. Therefore, we conclude that u is a smooth function in $\mathbb{R}^3 \times (0, T]$.

This conclusion contradicts with our assumption that u would cease to be regular at time T for the first time. This contradiction implies that u can not develop a local self-similar singularity in any finite time. This completes the proof of Theorem 1.

Appendix I.

In this appendix, we prove Lemma 3.

Proof of Lemma 3. It is sufficient to obtain an upper bound for the recurrence equalities

$$\widetilde{K}_{n+1} = \tilde{K}_0 + M\tilde{K}_n^2, \qquad \tilde{K}_0 = K_0. \tag{2.62}$$

It is easy to see that $K_n \leqslant \tilde{K}_n$, for all $n \geqslant 1$. To simplify the notation, we will drop the tilde in $\tilde{K}_n$ in the following. Define $l_n = K_{n+1} - K_n$, then we have

$$l_n = M\left(K_{n-1} + K_n\right) l_{n-1}. \tag{2.63}$$

It is easy to see that $l_n > 0$ for all $n \geqslant 0$ and K_n is a monotonely increasing sequence. We claim that

$$M\left(K_{j-1} + K_j\right) \leqslant \frac{1}{2}, \qquad \text{for all } \ j \geqslant 1. \tag{2.64}$$

We will prove (2.64) by an induction argument.

1. For $j = 1$, we have

   $$M\left(K_0 + K_1\right) = M\left(K_0 + K_0 + MK_0^2\right) \leqslant \frac{1}{2} \tag{2.65}$$

   from the assumption $K_0 M \leqslant \frac{1}{6}$.

2. Assume that (2.64) holds for all $j \leqslant n$, we will prove that it also hold for $j = n+1$. Let $\alpha = 1/2$. It follows from (2.63) and the induction assumption that

   $$l_j \leqslant \alpha l_{j-1}, \quad \text{for all } \ 1 \leqslant j \leqslant n, \tag{2.66}$$

   which implies that

   $$l_j \leqslant \alpha^j l_0. \tag{2.67}$$

Applying $K_{n+1} = K_n + l_n$ recursively and using (2.67), we obtain

$$\begin{aligned} K_{n+1} &= K_0 + \textstyle\sum_{j=0}^{n} l_j \\ &\leqslant K_0 + l_0 \textstyle\sum_{j=0}^{n} \alpha^j \\ &= K_0 + l_0 \frac{1-\alpha^{n+1}}{1-\alpha} \\ &\leqslant K_0 + 2MK_0^2 \leqslant \tfrac{4}{3} K_0, \end{aligned} \tag{2.68}$$

where we have used $MK_0 \leqslant 1/6$. Define $K_{\max} = \frac{4}{3}K_0$. Then we have

$$2MK_{\max} = \frac{8}{3}MK_0 \leqslant \frac{4}{9} < \frac{1}{2}. \tag{2.69}$$

Thus, we obtain

$$M\left(K_n + K_{n+1}\right) \leqslant 2MK_{\max} < \frac{1}{2}. \tag{2.70}$$

This proves the claim (2.64) by induction, and we obtain

$$K_n \leqslant K_{\max}, \quad \text{for all } \ n \geqslant 0. \tag{2.71}$$

We have already shown that $2MK_{\max} < \dfrac{1}{2}$ in (2.69). This completes the proof of Lemma 3.

Appendix II. Proof of estimate (2.39)

In this appendix, we prove estimate (2.39). First, we state a useful inequality

$$|1 - e^{-2x}| \geqslant (1 - e^{-2})|x|, \quad \text{for } \ 0 \leqslant x \leqslant 1, \tag{2.72}$$

which is a consequence of the fact that $(1 - e^{-x})/x$ is a monotonely decreasing function for $x > 0$. We consider two cases. If $\tau > 1$, we divide the integral into two parts as follows:

$$\begin{aligned} & \textstyle\int_0^{\tau} e^{\gamma s} \left(1 - e^{-2(\tau - s)}\right)^{-\frac{1+\gamma}{2}} ds \\ = & \textstyle\int_0^{\tau-1} + \int_{\tau-1}^{\tau} e^{\gamma s} \left(1 - e^{-2(\tau - s)}\right)^{-\frac{1+\gamma}{2}} ds \\ \leqslant & \textstyle\int_0^{\tau-1} e^{\gamma s} \left(1 - e^{-2}\right)^{-\frac{1+\gamma}{2}} ds + \int_{\tau-1}^{\tau} e^{\gamma \tau} \left((1 - e^{-2})(\tau - s)\right)^{-\frac{1+\gamma}{2}} ds \\ = & \left(1 - e^{-2}\right)^{-\frac{1+\gamma}{2}} \tfrac{e^{\gamma(\tau-1)} - 1}{\gamma} - e^{\gamma\tau} \left(1 - e^{-2}\right)^{-\frac{1+\gamma}{2}} \tfrac{2}{1-\gamma} \left(\tau - s\right)^{\frac{1-\gamma}{2}} \Big|_{s=\tau-1}^{s=\tau} \\ = & \left(1 - e^{-2}\right)^{-\frac{1+\gamma}{2}} \tfrac{e^{\gamma(\tau-1)} - 1}{\gamma} + \tfrac{2}{1-\gamma} e^{\gamma\tau} \left(1 - e^{-2}\right)^{-\frac{1+\gamma}{2}} \\ \leqslant & \left(\tfrac{2}{1-\gamma} + \tfrac{1}{\gamma}\right) \left(1 - e^{-2}\right)^{-\frac{1+\gamma}{2}} e^{\gamma\tau}, \end{aligned} \tag{2.73}$$

where we have used (2.72). Thus we prove

$$e^{-\tau}\int_0^\tau e^{\gamma s}\left(1-e^{-2(\tau-s)}\right)^{-\frac{1+\gamma}{2}}ds\leqslant c_1e^{-(1-\gamma)\tau}<c_1,\qquad \text{for all }\ \tau>1,$$

where c_1 is defined in (2.13). For $\tau\leqslant 1$, we have by using (2.72)

$$\begin{aligned}
e^{-\tau}\int_0^\tau e^{\gamma s}\left(1-e^{-2(\tau-s)}\right)^{-\frac{1+\gamma}{2}}ds &\leqslant e^{-(1-\gamma)\tau}\int_0^\tau\left(\left(1-e^{-2}\right)(\tau-s)\right)^{-\frac{1+\gamma}{2}}ds\\
&\leqslant \frac{2e^{-(1-\gamma)\tau}}{1-\gamma}\left(1-e^{-2}\right)^{-\frac{1+\gamma}{2}}\tau^{\frac{1-\gamma}{2}}\\
&\leqslant c_1\tau^{\frac{1-\gamma}{2}}e^{-(1-\gamma)\tau}\leqslant c_1. \qquad (2.74)
\end{aligned}$$

This proves (2.39).

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